A research problem on

"Orthogonal one- and two-tailed

multiple comparisons with equal individual error rates".

D. S. Robson

BU-387-M

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## ABSTRACT

When factorial treatment designs are used in experimentation the investigator frequently has prior knowledge concerning some or all of the main effects. One-tailed t-tests of the significance of main effect contrasts which are known in advance to be non-negative (or non-positive) increases the power of these individual tests and hence increases the power of the corresponding multiple comparison. In order to control the (experiment-wide) error rate for the simultaneous testing of m orthogonal contrasts, k of which are known in advance to be non-negative, we are required to calculate critical values $t_1$ and $t_2$ such that

$$1-\alpha = \int_0^\infty \left[\Phi\left(t_1\sqrt{\frac{u}{\nu}}\right)\right]^k \left[1-2\Phi\left(-t_2\sqrt{\frac{u}{\nu}}\right)\right]^{m-k} f_{\chi^2_\nu}(u)\, du$$

where $f_{\chi^2_\nu}$ is the chi-square density function on $\nu$ error degrees of freedom. Uniqueness can be achieved by requiring that the individual (marginal) error rates of the m contrasts be constant.

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In experiments utilizing factorial treatment designs the direction of some or all main effects is frequently known a priori. Previous experience may have shown that increasing the level of factor A can only increase yield, regardless of the levels of other factors; similarly, increasing the level of B might be known to decrease yield regardless of the level of all other factors. Thus, in a $2^k$ factorial experiment the direction (sign) of some or all of the k main effects might be known in advance, and in such situations the power of the simultaneous test of all $2^k-1$ single degree of freedom contrasts can be increased by exploiting this prior knowledge through the use of one-tailed rejection criteria on those contrasts of known sign. We are thus led to consider the special multiple comparison problem where a null hypothesis specifies that the chance variables $Y_1,\ldots,Y_m$ are NIID$(0,\sigma^2)$ and $s^2$ is an estimate of $\sigma^2$ with $\nu s^2/\sigma^2$ distributed as $\chi^2_\nu$ independently of $Y_1,\ldots,Y_m$, while the alternative hypothesis specifies that $\mu_1 > 0,\ldots,\mu_k > 0$, $\mu_{k+1} \neq 0,\ldots,\mu_m \neq 0$. Since only positive values of $\mu_1,\ldots,\mu_k$ are admitted into this alternative hypothesis the marginal power of these k individual tests will be maximized by using one-sided rejection regions, while two-sided rejection regions must be used for the remaining m-k individual tests.

The probability of accepting the null hypothesis $H_0 : \mu_1 = \ldots = \mu_m = 0$ when this hypothesis is true is simply

$$1-\alpha = P_{H_0}\{Y_1 < t_1 s,\ldots,Y_k < t_1 s \text{ and } |Y_{k+1}| < t_2 s,\ldots,|Y_m| < t_2 s\}$$

and the marginal, individual error rates are

$$\alpha_1 = P_{H_o}(Y_i > t_1 s) = P_{H_o}(Y_i < -t_1 s) \quad \text{for} \quad i = 1,\ldots,k$$

$$\alpha_2 = 2P_{H_o}(Y_i < -t_2 s) \quad \text{for} \quad i = k+1,\ldots,m$$

Since $Y_i/s$ is distributed as Student's t on $\nu$ degrees of freedom when $H_o$ is true then

$$\alpha_1 = F_{T_\nu}(-t_1) \qquad \alpha_2 = 2F_{T_\nu}(-t_2)$$

where $F_{T_\nu}$ is the cumulative distribution function of Student's t on $\nu$ d.f.. The constraint of equal marginal error rates, $\alpha_1 = \alpha_2$, then uniquely determines $t_2$ as a function of $t_1$,

$$-t_2 = F_{T_\nu}^{-1}\left(\tfrac{1}{2}F_{T_\nu}(-t_1)\right). \tag{1}$$

For a fixed value of s the conditional probability of accepting $H_o$ is

$$P_{H_o}\{Y_1 < t_1 s,\ldots,Y_k < t_1 s,\ |Y_{k+1}| < t_2 s,\ldots,\ |Y_m| < t_2 s \ \big|\ s\}$$

$$= P_{H_o}(Y_1 < t_1 s|s)\ldots P_{H_o}(Y_k < t_1 s|s)P_{H_o}(-t_2 s < Y_{k+1} < t_2 s|s)\ldots P_{H_o}(-t_2 s < Y_m < t_2 s|s)$$

$$= \left[\Phi\left(\frac{t_1 s}{\sigma}\right)\right]^k \left[1-2\Phi\left(-\frac{t_2 s}{\sigma}\right)\right]^{m-k}$$

Since $\nu s^2/\sigma^2$ is distributed as $\chi^2_\nu$ then letting $s/\sigma = \sqrt{u/\nu}$ we obtain the unconditional probability as

$$1-\alpha = \int_0^\infty \left[\Phi\left(t_1\sqrt{\frac{u}{\nu}}\right)\right]^k \left[1-2\Phi\left(-t_2\sqrt{\frac{u}{\nu}}\right)\right]^{m-k} f_{\chi^2_\nu}(u)\, du \tag{2}$$

where $f_{\chi^2_\nu}$ is the chi-square density function on $\nu$ degrees of freedom. The calculation of (2) subject to the constraint (1) will thus lead to useful tables of critical values $(t_1,t_2)$ for a range of values of $m,k,\nu$ and $\alpha$.

Certain limiting cases of this problem have already been solved; at $\nu = \infty$ the constraint (1) becomes

$$\Phi(-t_1) = 2\Phi(-t_2) \quad \text{or} \quad \Phi(t_1) = 1-2\Phi(-t_2)$$

while (2) becomes

$$1-\alpha = \left[\Phi(t_1)\right]^k \left[1-2\Phi(-t_2)\right]^{m-k} = \left[\Phi(t_1)\right]^m$$

$$t_1 = \Phi^{-1}\left((1-\alpha)^{\frac{1}{m}}\right)$$

$$t_2 = -\Phi^{-1}\left(\frac{1-(1-\alpha)^{\frac{1}{m}}}{2}\right)$$

The limiting case $k = 0$ has been tabulated and provides a convenient bound on $t_2$ (see, for example, Miller's book on multiple comparisons) for specific values of $m, \nu$ and $\alpha$. A numerical solution for the case $k = m$ would likewise provide bounds on $t_1$, and the solution to equation (1) could then be calculated over the range required for intermediate values of $k$, $0 < k < m$. Once the required locus of (1) is available then (2) can be solved iteratively by numerical integration.